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On the Numerical Solution of Difficult Boundary-Value Problems

A. DAVEY

School of Mathematics, University of Newcastle upon Tyne, Newcastle upon Tyne NE1 7RU, England

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We present a simple explanation of how to use the compound matrix method to solve difficult linear inhomogeneous two-point boundary-value problems and we discuss in detail the solution of two standard problems. We also give numerical results for an Orr–Sommerfeld problem which illustrate the efficacy of the compound matrix method compared to orthonormalization.

1. INTRODUCTION

In this paper we are concerned with the numerical solution of linear inhomogeneous two-point boundary-value problems for systems of ordinary differential equations by the use of explicit shooting methods. We shall restrict our attention to "difficult" problems, by which we mean "stiff" differential systems for which the real parts of the characteristic values of the differential operator will be widely separated. In a fundamental paper Conte [1] has clearly explained the pitfalls which may be encountered if one attempts to solve such a difficult problem by using the standard (superposition) shooting method, and there is a considerable literature devoted to devising other shooting methods which obviate these pitfalls. Perhaps the most popular of these other methods is that of orthonormalization due to Godunov [5], a concise account of which is given in Conte's paper. Another method which has recently received a great deal of attention is the Riccati method, see for example Scott [9]. Both the orthonormalization method and the Riccati method do however have their disadvantages, with orthonormalization it is a laborious accounting feat to construct the required function, and the differential equations of the Riccati method have annoying singularities.

More than twelve years ago Gilbert and Backus [4] advocated the use of compound matrices for solving difficult linear eigenvalue problems but until very recently their paper seems to have been completely overlooked by both numerical analysts and other applied mathematicians. It now transpires that their method is both very important and mathematically interesting as evidenced by Ng and Reid [6] who recently rediscovered and used the method to solve some difficult eigenvalue problems. See also [3] for an explanation as to why the method is so successful.

In Section 2 of this paper we explain how to use the compound matrix method to solve difficult linear inhomogeneous two-point boundary-value problems. We do

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this by presenting full details of the method for solving the general fourth-order differential equation, and, where appropriate, we indicate in the text how to proceed for any other specific problem. We also give two numerical examples of standard fourth-order problems. Then, in Section 3, we present a numerical comparison of the solution of the Orr–Sommerfeld equation for plane Poiseuille flow at high Reynolds numbers using the compound matrix method and using orthonormalization.

2. The Compound Matrix Method for Inhomogeneous Problems

The general linear inhomogeneous problem may be expressed in the form $\boldsymbol{\phi}' = \mathbf{A}\boldsymbol{\phi} + \mathbf{r}$, where $\boldsymbol{\phi}$, $\mathbf{r}$ are n-vectors and $\mathbf{A}$ is an $n \times n$ matrix. We suppose that the boundary conditions are separated so that q boundary conditions are known at one end of the range of integration and p at the other end; $p + q = n$. Let $q \geqslant p$ then we will call the q conditions the *initial* conditions so that there will be p *unknown* initial conditions, and the integration will be from the q end to the p end. For this problem $\boldsymbol{\phi}$ can always be suitably redefined by a non-singular transformation so that the initial conditions on $\boldsymbol{\phi}$ simply become that the first q components of $\boldsymbol{\phi}$ are specified.

Instead of presenting the general theory, for the sake of clarity we discuss in detail a case with $n = 4$ and $p = 2$, *but* in the *text* which follows we retain the use of n, p rather than 4, 2 so that the reader will know what to do for any other problem. (The general theory for the homogeneous problem may be gleaned by reading Ng and Reid [6] in conjunction with Schwarz [8].)

In order to solve a linear inhomogeneous problem most methods obtain the solution by forming an appropriate combination of the solutions of the associated homogeneous problem with a particular integral and the compound matrix method is no exception to this general rule. Since we shall need to discuss therefore the associated homogeneous problem we follow the notation used by Ng and Reid [6] and like them we illustrate the ideas involved by considering a simple, but sufficiently general, example of the fourth-order two-point boundary-value problem

$$L\phi \equiv \phi'''' - a_1\phi''' - a_2\phi'' - a_3\phi' - a_4\phi = r, \tag{1}$$

where a $'$ denotes differentiation with respect to the independent variable x and a_1, a_2, a_3, a_4 and r are known functions of x. Although the boundary conditions and the range of integration may be quite general, to be definite we suppose that the initial conditions are $\phi = \beta$, $\phi' = \gamma$ when $x = 0$ and that the range of integration is $0 \leqslant x \leqslant 1$. There will also be two boundary conditions at $x = 1$ but we do not need to specify these until later. By the associated homogeneous problem to (1) we shall mean $L\phi = 0$ with $\phi = \phi' = 0$ when $x = 0$.

Firstly, let ϕ_1, ϕ_2 be any two linearly independent solutions of the associated homogeneous problem $L\phi = 0$ which satisfy the initial conditions $\phi = \phi' = 0$ at $x = 0$ and consider the $n \times p$ solution matrix

$$\mathbf{\Phi} = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \\ \phi_1''' & \phi_2''' \end{pmatrix}. \tag{2}$$

The $p \times p$ minors of $\mathbf{\Phi}$ are, in lexical order,

$$\begin{aligned} y_1 &= \phi_1\phi_2' - \phi_1'\phi_2, & y_4 &= \phi_1'\phi_2'' - \phi_1''\phi_2', \\ y_2 &= \phi_1\phi_2'' - \phi_1''\phi_2, & y_5 &= \phi_1'\phi_2''' - \phi_1'''\phi_2', \\ y_3 &= \phi_1\phi_2''' - \phi_1'''\phi_2, & y_6 &= \phi_1''\phi_2''' - \phi_1'''\phi_2'', \end{aligned} \tag{3}$$

and since they are the Plucker coordinates of the line joining $(\phi_1, \phi_1', \phi_1'', \phi_1''')^T$ and $(\phi_2, \phi_2', \phi_2'', \phi_2''')^T$ in three-dimensional projective space the point $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5, y_6)^T$ lies on the Klein quadric, see [10], and so

$$y_1y_6 - y_2y_5 + y_3y_4 = 0. \tag{4}$$

The closed differential system for the components of $\mathbf{y}$ may be found by differentiating (3) and using the homogeneous form $L\phi = 0$ of (1) and it is the *linear* system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3 + y_4, \\ y_3' &= a_3y_1 + a_2y_2 + a_1y_3 + y_5, \\ y_4' &= y_5, \\ y_5' &= -a_4y_1 + a_2y_4 + a_1y_5 + y_6, \\ y_6' &= -a_4y_2 - a_3y_4 + a_1y_6. \end{aligned} \tag{5}$$

The null initial conditions on ϕ_1, ϕ_2 at $x = 0$ translate to $y_1 = y_2 = y_3 = y_4 = y_5 = 0$ and we may set $y_6 = 1$ because (5) is homogeneous. The numerical solution of (5) subject to these conditions may be found by a straightforward Runge–Kutta integration, a subdominant solution of this *initial* value problem is not required.

Second let ψ be any particular solution of the inhomogeneous problem (1) which satisfies the known initial conditions $\psi(0) = \beta$, $\psi'(0) = \gamma$. It is important *not* to calculate such a ψ directly but to proceed instead by solving, simultaneously with (5), the fourth-order differential system for the $p + 1 \times p + 1$ minors of the matrix

$$\mathbf{\Psi} = \begin{pmatrix} \psi & \phi_1 & \phi_2 \\ \psi' & \phi_1 & \phi_2' \\ \psi'' & \phi_1'' & \phi_2'' \\ \psi''' & \phi_1''' & \phi_2''' \end{pmatrix}, \tag{6}$$

namely

$$z_1 = \begin{vmatrix} \psi & \phi_1 & \phi_2 \\ \psi' & \phi_1' & \phi_2' \\ \psi'' & \phi_1'' & \phi_2'' \end{vmatrix}, \qquad z_2 = \begin{vmatrix} \psi & \phi_1 & \phi_2 \\ \psi' & \phi_1' & \phi_2' \\ \psi''' & \phi_1''' & \phi_2''' \end{vmatrix},$$

$$z_3 = \begin{vmatrix} \psi & \phi_1 & \phi_2 \\ \psi'' & \phi_1'' & \phi_2'' \\ \psi''' & \phi_1''' & \phi_2''' \end{vmatrix}, \qquad z_4 = \begin{vmatrix} \psi' & \phi_1' & \phi_2' \\ \psi'' & \phi_1'' & \phi_2'' \\ \psi''' & \phi_1''' & \phi_2''' \end{vmatrix}. \tag{7}$$

By differentiating (7) and using $L\psi = r$, $L\phi_1 = L\phi_2 = 0$ and (3) we readily obtain the inhomogeneous linear system

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= a_2 z_1 + a_1 z_2 + z_3 + r y_1, \\ z_3' &= -a_3 z_1 + a_1 z_3 + z_4 + r y_2, \\ z_4' &= a_4 z_1 + a_1 z_4 + r y_4, \end{aligned} \tag{8}$$

to integrate at the same time as (5). The initial conditions at $x = 0$ for z_1, z_2, z_3, z_4 may be found from the definitions (7) and they are

$$(z_1, z_2, z_3, z_4)^T = (0, 0, \beta, \gamma)^T. \tag{9}$$

Having found $y_1 - y_6$, $z_1 - z_4$ by the method described above we may now find the required solution ϕ of the originally posed problem (1) by reasoning that there must exist constants λ, μ such that

$$\phi = \psi + \lambda\phi_1 + \mu\phi_2,$$

so that also

$$\begin{aligned} \phi' &= \psi' + \lambda\phi_1' + \mu\phi_2', \\ \phi'' &= \psi'' + \lambda\phi_1'' + \mu\phi_2'', \end{aligned} \tag{10}$$

and

$$\phi''' = \psi''' + \lambda\phi_1''' + \mu\phi_2'''.$$

The constants λ and μ can be eliminated from (10) in $\binom{n}{p+1}$ different ways and if this is done then we obtain

$$\begin{aligned} y_1\phi'' - y_2\phi' + y_4\phi &= z_1, \\ y_1\phi''' - y_3\phi' + y_5\phi &= z_2, \\ y_2\phi''' - y_3\phi'' + y_6\phi &= z_3, \\ y_4\phi''' - y_5\phi'' + y_6\phi' &= z_4, \end{aligned} \tag{11}$$

and we must choose one of these $\binom{n}{p+1}$ equations to determine the required solution ϕ. For a good discussion of which equation to use for eigenvalue problems when $z_1 = z_2 = z_3 = z_4 = 0$ see Ng and Reid [6], in practice it is simplest to try them all in turn for a few test cases and see which gives the most accurate solution near $x = 0$ for a specific problem.

It is only now that we need the boundary conditions on ϕ at $x = 1$; after $y_1 - y_6$, $z_1 - z_4$ have been found then one of (11) is integrated backwards from $x = 1$ to $x = 0$ using the boundary conditions on ϕ at $x = 1$ to start the integration. Usually, the first of (11), which is only of second order, will be the appropriate equation to use, especially if the boundary conditions on ϕ at $x = 1$ are of the form $\phi = \delta$, $\phi' = \epsilon$ where δ, ϵ are constants. The equations for $y_1 - y_6$, $z_1 - z_4$ are numerically unstable during a backward integration and so their values should be stored during the forward integration, or at least every so often. Hence during the backward integration if the equations for $y_1 - y_6$, $z_1 - z_4$ are integrated simultaneously with one of (11) then their values can be reset to the stored values when necessary. Note that (11) cannot be integrated right up to $x = 0$ because the coefficients of the highest derivatives of ϕ are zero there, in practice however sufficient information about ϕ and its derivatives at $x = 0$ may be gleaned from either the boundary conditions on ϕ at $x = 0$ or by extrapolation for small values of x.

The success of the compound matrix method, see [3], lies basically in the fact that all the coefficients in (11), such as $y_1 = \phi_1\phi_2' - \phi_1'\phi_2$ for example, are calculated *directly* rather than as a combination of calculated quantities, the evaluation of $y_1 - y_6$, $z_1 - z_4$ is rather like an automatic orthonormalization. To illustrate the power of the method we shall now use it to solve two standard examples of difficult linear inhomogeneous two-point boundary-value problems.

As our first example we consider a problem posed by Conte [1], (see Example 2 on page 316 of his paper), which has the merit that it has a known exact solution. The problem is

$$\phi'''' - 3601\phi'' + 3600\phi = -1 + 1800x^2, \tag{12}$$

with boundary conditions

$$\phi(0) = 1, \qquad \phi'(0) = 1, \tag{13}$$

$$\phi(1) = 3/2 + \sinh 1, \qquad \phi'(1) = 1 + \cosh 1. \tag{14}$$

The exact solution of this problem is

$$\phi = 1 + \frac{x^2}{2} + \sinh x, \tag{15}$$

however the characteristic values of the differential operator in (12) are ± 1 and ± 60 and so the posed problem is numerically "stiff" because any numerical solution of (12) will contain a component proportional to $\exp(60x)$ even if only due to round-off errors. Using Runge–Kutta integration and a step-length $h = 0.001$ with double-

TABLE I

The Errors Given by the Standard Shooting Method and by the Compound Matrix Method for Conte's Problem

x	Error $\times 10^7$ using standard superposition and $h = 0.001$	Error $\times 10^7$ using compound matrix method $h = 0.001$	$h = 0.0001$
0.0	0	0.0	0.00000
0.1	4,472	0.4	0.00004
0.2	9,878	0.8	0.00009
0.3	15,388	1.2	0.00013
0.4	22,384	1.5	0.00016
0.5	567,127	1.7	0.00018
0.6	2×10^8	1.8	0.00019
0.7	9×10^{10}	1.9	0.00020
0.8	4×10^{13}	1.7	0.00018
0.9	10^{16}	1.3	0.00013
1.0	0	0.0	0.00000

precision (almost 17 decimal places) Fortran on an IBM 370/168 we applied the standard (superposition) shooting method to (12)–(14) and, as expected, we obtained completely nonsensical results, except for small values of x, as shown in Table I.

We then used the compound matrix method as described above in (12)–(14), the relevant equations for $y_1 - y_6$ are

$$
\begin{aligned}
y_1' &= y_2 , \\
y_2' &= y_3 + y_4 , \\
y_3' &= a_2 y_2 + y_5 , \\
y_4' &= y_5 , \\
y_5' &= -a_4 y_1 + a_2 y_4 + y_6 , \\
y_6' &= -a_4 y_2 ,
\end{aligned} \tag{16}
$$

where $a_2 = 3601$ and $a_4 = -3600$. From the homogeneous form of (13) the initial conditions at $x = 0$ are $y_1 = y_2 = y_3 = y_4 = y_5 = 0$, $y_6 = 1$. The equations for $z_1 - z_4$ are, with $r = -1 + 1800x^2$,

$$
\begin{aligned}
z_1' &= z_2 , \\
z_2' &= a_2 z_1 + z_3 + r y_1 , \\
z_3' &= z_4 + r y_2 , \\
z_4' &= a_4 z_1 + r y_4 ,
\end{aligned} \tag{17}
$$

and from (7), (13) the initial conditions are $z_1 = z_2 = 0$, $z_3 = z_4 = 1$ when $x = 0$. To determine ϕ we use the first of (11) namely

$$y_1\phi'' - y_2\phi' + y_4\phi = z_1\,, \tag{18}$$

especially since we know ϕ, ϕ' when $x = 1$ from (14). Incidentally for this particular problem notice that (17), (18) do not involve y_3, y_5, y_6 and it just so happens that these three quantities can easily be eliminated from (16) to yield

$$\begin{aligned} y_1' &= y_2\,, \\ y_2' &= a_2 y_1 + 2y_4\,, \\ y_4'' &= -2a_4 y_1 + a_2 y_4 + 1, \end{aligned} \tag{19}$$

hence reducing the differential order by 2; the initial conditions for (19) when $x = 0$ are $y_1 = y_2 = y_4 = y_4' = 0$. A Runge–Kutta method was again used with $h = 0.001$ and the results are shown in Table I, there is a dramatic improvement, the largest error now being less than 2.10^{-7}, and this is reduced to 2.10^{-11} when $h = 0.0001$ as also shown in Table I. (Although (18) cannot be integrated quite up to $x = 0$, because $y_1 = 0$ there, we show the error at $x = 0$ in Table I as being zero because we know the value of ϕ at $x = 0$ from (13).)

For our second example we consider a fundamental inhomogeneous problem taken from hydrodynamic stability theory, in brief the calculation of the harmonic function of non-linear stability theory at the nose of the neutral stability curve for plane Poiseuille flow. The mathematical description of the problem is

$$\{D^2 - \alpha^2 - i\alpha R(1 - x^2 - c)\}\{D^2 - \alpha^2\}E - 2i\alpha RE = 0, \tag{20}$$

$$E' = E''' = 0 \text{ when } x = 0, \qquad E = E' = 0 \text{ when } x = 1, \tag{21}$$

$$\{D^2 - 4\alpha^2 - 2i\alpha R(1 - x^2 - c)\}\{D^2 - 4\alpha^2\}\phi - 4i\alpha R\phi = i\alpha R(E'E'' - EE'''), \tag{22}$$

$$\phi = \phi'' = 0 \text{ when } x = 0, \qquad \phi = \phi' = 0 \text{ when } x = 1. \tag{23}$$

In (20)–(23) both D and a $'$ denote differentiation with respect to x, $\alpha = 1.02055$ is the wavenumber, $R = 5772.222$ is the Reynolds number, $c = 0.26400055$ is the wave-speed of the disturbance, E is the eigenfunction solution of the Orr–Sommerfeld equation (20), and ϕ is the required solution of the inhomogeneous equation (22) with boundary conditions (23). Thus a necessary preliminary to calculating ϕ is first to solve the eigenvalue problem (20), (21) and find the eigenvalue c and the eigenfunction E; this has been done using the compound matrix method by Ng and Reid [6], although for different values of α and R. For the values of α and R given above c, E cannot be found by the standard shooting method using double-precision Fortran.

The computer program which we wrote first used the compound matrix method to solve the Orr–Sommerfeld problem (20), (21) and it stored the values of c and of E

and its derivatives. Second it found $y_1 - y_6$, $z_1 - z_4$ for the inhomogeneous problem (22), (23) by solving (16), (17) with

$$\begin{aligned} a_2 &= 8\alpha^2 + 2i\alpha R(1 - x^2 - c), \\ -a_4 &= 4\alpha^2\{4\alpha^2 + 2i\alpha R(1 - x^2 - c)\} - 4i\alpha R, \\ r &= i\alpha R(E'E'' - EE'''), \end{aligned} \tag{24}$$

since (22) is of the same form as (12). The boundary conditions (23) when $x = 0$ imply that there all of $y_1 - y_6$, $z_1 - z_4$ are zero except y_5 which may be set to 1. Third the program then integrated (18) backwards from $x = 1$ until nearly $x = 0$, using previously stored values of $y_1 - y_6$, $z_1 - z_4$, and hence determined the required solution ϕ, the value of ϕ at $x = 0$ is of course zero from (23).

Three runs were done with step-lengths h of 0.002, 0.001 and 0.0005 and these results were extrapolated to yield the values of ϕ shown in Table II. Also the real and imaginary parts of ϕ are plotted in Fig. 1 and this should be compared with Fig. 3e of Reynolds and Potter [7] who originally solved (22), (23) for very similar values of α and R. Because of the rather unusual way in which Reynolds and Potter defined their Reynolds number there is a scale factor difference of 3/2 between our results and theirs but otherwise the two figures are virtually identical.

TABLE II

Some of the Values Obtained for the Solution of the Harmonic Function ϕ of Our Second example, Eqs. (20)–(23)

x	ϕ
0.0	0
0.1	0.711–0.107i
0.2	1.426–0.217i
0.3	2.147–0.329i
0.4	2.874–0.444i
0.5	3.586–0.561i
0.6	4.242–0.677i
0.7	4.727–0.765i
0.8	4.705–0.978i
0.9	3.306–0.847i
1.0	0

The results of the two examples discussed above clearly illustrate the effectiveness of the compound matrix method. Although both examples have rather special differential equations and boundary conditions, as we mentioned earlier in this section the method can be used for the general linear inhomogeneous problem with separated boundary conditions.

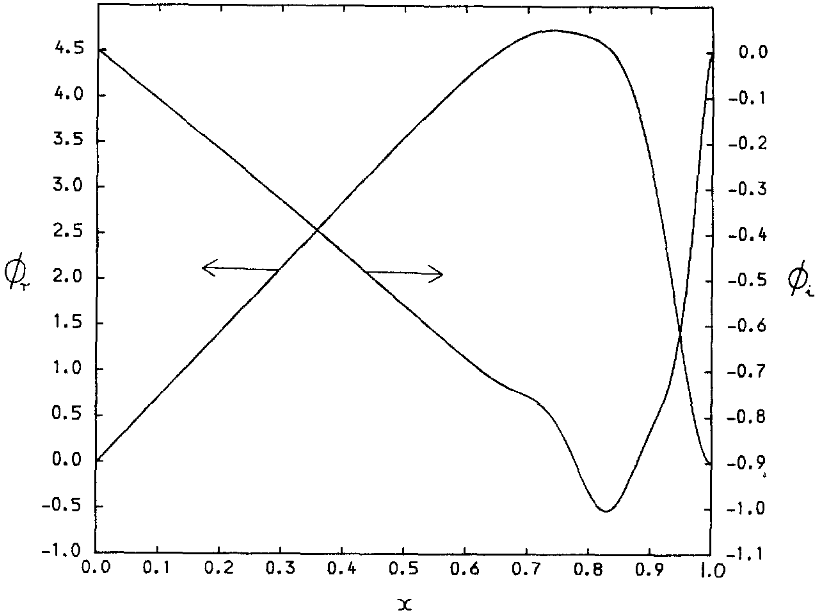

FIG. 1. The real and imaginary parts ϕ_r, ϕ_i of the solution for the harmonic function ϕ of our second example, see equations (20)–(23); apart from the scale factor of 3/2 (see text) it is virtually identical to the earlier calculation by Reynolds & Potter [7], see Fig. 3e of their paper.

3. Comparison of the Compound Matrix Method with Orthonormalization

The most important advantage of using the compound matrix method for difficult problems relative to other shooting methods is that it can be used by someone who only needs to know how to use a Runge–Kutta integration routine, all other shooting methods for difficult problems require much more knowledge. We do not pretend that it is as efficient as other methods, nevertheless we feel that at least we should compare its efficiency with that of orthonormalization. In order to make this comparison we now use both methods to calculate the eigenvalue c which corresponds to the most unstable mode in the classical linear stability problem of plane Poiseuille flow, when the wavenumber $\alpha = 1$ and the Reynolds number R is large. The differential equation for this problem is the Orr–Sommerfeld equation (20) with boundary conditions (21) and the characteristic values of the differential operator are of order ± 1 and $\pm R^{1/2}$. To obtain a true comparison between using the compound matrix method and using orthonormalization we shall be particularly concerned with solutions for very large values of R; for a similar comparison between the Riccati method and orthonormalization see [2].

Firstly we use the orthonormalization method together with a Runge–Kutta routine to integrate (20), (21) from $x = 0$ to $x = 1$ and we iterate to the eigenvalue c by a Newton–Raphson process, until two successive iterates of c differ by less than some small preassigned error tolerance. This error tolerance is chosen to be so small that the value of c obtained depends predominantly upon the number of integration steps used rather than on the tolerance requirement. For $\log R = 6(1)10$ we determine the least number of integration steps of equal length which we can use to calculate the eigenvalue c correct to 4 significant figures. When R is very large this number is of order $R^{1/2}$ since the principal restriction is that the Runge–Kutta routine shall be convergent.

Second we repeat these calculations using the compound matrix method and since we are only interested in obtaining the eigenvalue c it suffices to solve (16) with

$$a_2 = 2 + iR(1 - x^2 - c),$$

and

$$a_4 = -1 + iR(1 + x^2 + c). \tag{25}$$

The initial conditions are, from (21), that $y_1 - y_6$ are all zero except $y_2 = 1$ and c must be iterated upon to satisfy the condition that y_1 should be zero when $x = 1$.

The comparison found between the two methods is shown in Table III which contains the eigenvalue c correct to 8 decimal places. The column headed ONIZ is approximately the least number of integration steps which may be used to calculate c correct to 4 significant figures using orthonormalization. The coulmn headed CMM is the corresponding number via the compound matrix method. (For the corresponding numbers when the Riccati method is used see Table 1 of [2].)

TABLE III

The Number of Integration Steps of Equal Length Required by Orthonormalization and by the Compound Matrix Method to Calculate the Eigenvalue c, Correct to Four Significant Figures, of the Orr–Sommerfeld Problem (20), (21) for Plane Poiseuille Flow

$\log R$	c	ONIZ	CMM
6	$0.06659252–0.01398327i$	600	2,200
7	$0.03064130–0.00726049i$	1,200	4,200
8	$0.01417134–0.00351239i$	3,700	8,200
9	$0.00656630–0.00166002i$	12,000	33,000
10	$0.00304508–0.00077699i$	37,000	61,000

It is evident from Table III that when $R^{1/2} \geqslant 1000$, so that the characteristic values of the differential operator in (20) which are of order ± 1 and $\pm R^{1/2}$ are *very* widely separated, and hence the problem may be said to be *very* difficult, then the compound matrix method requires approximately between twice and four times as many integration steps as orthonormalization; (the Riccati method required approximately twice as many integration steps as orthonormalization). Both methods needed virtually the same number of iterations, usually three or four when the initial guess for the eigenvalue differed by about 0.1 % from the exact value.

The computing time required by the compound matrix method was found to be about twice as long as when using orthonormalization, we mentioned earlier that we did not expect the compound matrix method to be as fast as other shooting methods. In fact this timing ratio will be even larger for higher-order differential problems because for a differential system of order $2n$ then essentially the orthonormalization method only has to integrate $2n^2$ equations whereas the compound matrix method has to integrate $\binom{2n}{n}$ equations, assuming there are n boundary conditions at each end of the range of integration.

Of course really both methods should have been used with sophisticated variable-step integration routines to accommodate the structures of E and of $y_1 - y_6$. However the integral curves associated with the two methods are very similar and so if this were done there would be little change in the *ratio* of the number of integration steps or computing times needed by the two methods from those mentioned above.

4. Concluding Remarks

For linear homogeneous eigenvalue problems with separated boundary conditions the brilliancy of the compound matrix method is that it transforms difficult (i.e., stiff) two-point boundary-value problems which cannot be solved by the standard (superposition) shooting method into initial-value problems which can be solved by standard shooting because the required solutions will not be subdominant. The essence of the method is to determine the multilinear form (or wedge product) of the linearly independent solutions which satisfy the known initial conditions.

We have explained how to extend the use of the compound matrix method from eigenvalue problems to inhomogeneous two-point boundary-value problems. Although we have not given a detailed analysis for the general inhomogeneous problem the analysis which we have presented for a single fourth-order equation and the two examples which we have discussed should be sufficient to enable the reader to solve any particular problem in which he may be interested.

Also we have examined the efficiency of the compound matrix method compared to orthonormalization for a standard difficult eigenvalue problem and we found that the compound matrix method requires about three times as many integration steps and approximately twice as much computing time as orthonormalization. This comparison was for a fourth-order problem and for higher differential orders we expect the compound matrix method to require comparatively more computing time. However, the method has the important advantage that it is so easy to understand and program relative to other shooting methods for difficult problems, therefore it is an ideal method for use by those who are anxious to spend most of their time doing theoretical work and so wish to do their occassional computational work with the minimum of inconvenience.

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